# EQUIVALENCE OF THE DEFINITIONS OF 'CONTINUOUS'

**RECALL:** A function $f$ is said to be **continuous** at a point $x = a$ if $\lim_{x \to a} f(x) = f(a)$. More precisely:

Given $\epsilon > 0$ there is a $\delta > 0$ so that if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$. Said differently:

Given $\epsilon > 0$ there is a $\delta > 0$ so that if $x \in (a - \delta, a + \delta)$, then $f(x) \in (f(a) - \epsilon, f(a) + \epsilon)$. Said differently:

Given $\epsilon > 0$ there is a $\delta > 0$ so that $(a - \delta, a + \delta) \subseteq f^{-1}(f(a) - \epsilon, f(a) + \epsilon)$.

A function $f$ is said to be continuous on an interval if $f$ is continuous at each point in the interval.

**RECALL:** $(\mathbb{R}, \mathcal{E})$ denotes the real numbers with the Euclidean Topology.

That is $T \in \mathcal{E}$ iff for all $x \in T$, there is an interval $(a, b)$ so that $x \in (a, b) \subseteq T$.

**THEOREM:** A function $f : (\mathbb{R}, \mathcal{E}) \to (\mathbb{R}, \mathcal{E})$ is continuous iff $f^{-1}(U) \in \mathcal{E}$ for all $U \in \mathcal{E}$.

**PROOF:** ( $\Longrightarrow$ ) Suppose $f$ is continuous and suppose $U \in \mathcal{E}$. We need to show $f^{-1}(U) \in \mathcal{E}$.

That is, we need to show that if $x_0 \in f^{-1}(U)$, we can find an interval $(a, b)$ so that $x_0 \in (a, b) \subseteq f^{-1}(U)$.

Suppose $x_0 \in f^{-1}(U)$. Then $f(x_0) \in U$. Since $U \in \mathcal{E}$, there is an interval $(c, d)$ so that $f(x_0) \in (c, d) \subseteq U$.

Let $\epsilon = \min\{d - f(x_0), f(x_0) - c\}$. Then $(f(x_0) - \epsilon, f(x_0) + \epsilon) \subseteq (c, d) \subseteq U$.

Since $f$ is continuous, there exists $\delta > 0$ so that $(x_0 - \delta, x_0 + \delta) \subseteq f^{-1}(f(x_0) - \epsilon, f(x_0) + \epsilon) \subseteq f^{-1}(U)$.

Taking $(a, b) = (x_0 - \delta, x_0 + \delta)$, we have then $x_0 \in (a, b) \subseteq f^{-1}(U)$. Hence, we have shown $f^{-1}(U) \in \mathcal{E}$.

( $\Longleftarrow$ ): Suppose $f^{-1}(U) \in \mathcal{E}$ for all $U \in \mathcal{E}$. Let $a \in \mathbb{R}$ and $\epsilon > 0$.

You need to show you can find a $\delta > 0$ so that

$$(a - \delta, a + \delta) \subseteq f^{-1}(f(a) - \epsilon, f(a) + \epsilon)$$